Journal of Engineering Mathematics, Vol. 8, No. 2, April 1974
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Thermal stresses in heterogeneous anisotropic beams

Y.-R. KAN

Vehicle Research Department, General Motors Research Laboratories Warren, Michigan, USA

Y. M. ITO

Mechanics and Structures Department, University of California Los Angeles, California, USA

(Received July 31, 1973)

SUMMARY

A method is proposed for the exact solution of the two-dimensional thermoelastic equations for certain composite anisotropic beams by use of single Fourier series. Numerical examples demonstrating the usefulness of this method are presented.

1. Introduction

The thermal stresses and displacements of a homogeneous isotropic rectangular beam has been considered by Wah [1] by use of double Fourier series. Boley [2] and Boley and Tolins [3] have discussed the same problem by use of an infinite series of polynomial functions which are obtained by a differential recurrence relation. The latter method has also been used by Boley and Testa [4] to obtain the solution of a non-homogeneous but locally isotropic beam under arbitrary temperature distribution.

The present study gives the exact solution for the stresses and displacements of a free rectangular composite beam of length l, height h and small thickness δ which is under an arbitrary temperature distribution. The composite beam consists of n perfectly bonded layers with orthotropic thermal and mechanical material properties. The principal axes of orthotropy coincide with the beam axes.

The basic formulation requires that the second derivative of the temperature distribution with respect to the length has only a finite number of discontinuities in each layer. This assumption permits the formal expansion of the temperature distribution in a single Fourier series.

2. Thermoelasticity solution

Consider a composite beam composed of n orthotropic layers such that the various axes of material symmetry are parallel to the beam axes x, y, z. The beam occupies the region $0 \leqq x \leqq l$, $0 \leqq y \leqq h$ such that

$$(h/l) \ll 1 . \tag{1}$$

The thickness δ of the beam is assumed sufficiently small for the two-dimensional theory of plane stress to apply. Since each layer is orthotropic, the constitutive equations are given by [5]

$$\left.\begin{aligned} \varepsilon_x^{(i)} &= \frac{\sigma_x^{(i)}}{E_x^{(i)}} - \frac{\nu_{yx}^{(i)}}{E_y^{(i)}}\,\sigma_y^{(i)} + \alpha_x^{(i)} T^{(i)}, \qquad \varepsilon_y^{(i)} = \frac{\sigma_y^{(i)}}{E_y^{(i)}} - \frac{\nu_{xy}^{(i)}}{E_x^{(i)}}\,\sigma_x^{(i)} + \alpha_y^{(i)} T^{(i)} \\ \gamma_{xy}^{(i)} &= \frac{\tau_{xy}^{(i)}}{G_{xy}^{(i)}} \end{aligned}\right\} \tag{2}$$

where $E_x^{(i)}$, $E_y^{(i)}$, $\nu_{yx}^{(i)}$, $\nu_{xy}^{(i)}$, $G_{xy}^{(i)}$ are elastic constants such that $E_x^{(i)}\nu_{yx}^{(i)} = E_y^{(i)}\nu_{xy}^{(i)}$, $\alpha_x^{(i)}$, $\alpha_y^{(i)}$ are thermal expansion coefficients and $T^{(i)}$ is the temperature rise and the index i denotes the ith layer numbered from the bottom to the top of the beam. The strain displacement relations are

$$\varepsilon_x^{(i)} = u_{,x}^{(i)}, \quad \varepsilon_y^{(i)} = v_{,y}^{(i)}, \quad \gamma_{xy}^{(i)} = u_{,y}^{(i)} + v_{,x}^{(i)}, \tag{3}$$

where comma denotes differentiation.

If body forces are not considered, then the stress in each layer are determined by the formulas

$$\sigma_x^{(i)} = \phi_{,yy}^{(i)}, \quad \sigma_y^{(i)} = \phi_{,xx}^{(i)}, \quad \tau_{xy}^{(i)} = -\phi_{,xy}^{(i)}, \tag{4}$$

where the Airy stress function ϕ in two dimensions must satisfy the equation

$$\left(\frac{1}{E_x^{(i)}}\phi_{,yy}^{(i)}\right)_{,yy} - \left(\frac{v_{yx}^{(i)}}{E_y^{(i)}}\phi_{,xx}^{(i)}\right)_{,yy} + \left(\frac{1}{E_y^{(i)}}\phi_{,xx}^{(i)}\right)_{,xx} - \left(\frac{v_{xy}^{(i)}}{E_x^{(i)}}\phi_{,yy}^{(i)}\right)_{,xx} + \left(\frac{1}{G_{xy}^{(i)}}\phi_{xy}^{(i)}\right)_{,xy} = -(\alpha_y^{(i)}T^{(i)})_{,xx} - (\alpha_x^{(i)}T^{(i)})_{,yy}\,. \tag{5}$$

The stress free boundary conditions on the upper and lower surfaces are given by

$$\sigma_y(x, h) = \tau_{xy}(x, h) = 0\,, \quad \sigma_y(x, 0) = \tau_{xy}(x, 0) = 0 \tag{6}$$

It is not possible to satisfy exactly the conditions of zero tractions at the ends $x=0$, l, but only to satisfy the conditions that the tractions be self-equilibrating, namely that

$$\left.\begin{aligned} \delta\int_0^h \sigma_x(0, y)dy &= \delta\int_0^h \sigma_x(l, y)dy = 0 \\ \delta\int_0^h y\sigma_x(0, y)dy &= \delta\int_0^h y\sigma_x(l, y)dy = 0 \end{aligned}\right\} \tag{7}$$

$$\delta\int_0^h \tau_{xy}(0, y)dy = \delta\int_0^h \tau_{xy}(l, y)dy = 0\,. \tag{8}$$

Hence, condition (1) is necessary so as to insure a meaningful application of Saint-Venant's principle. Note that condition (8) is automatically satisfied when conditions (6) and (7) are imposed.

The perfect bonding of the layers requires the continuity of tractions and displacements at the interfaces, such that

$$\left.\begin{aligned} &\sigma_y^{(i)}(x, h_i) = \sigma_y^{(i+1)}(x, h_i)\,, \quad \sigma_{xy}^{(i)}(x, h_i) = \sigma_{xy}^{(i+1)}(x, h_i) \\ &u^{(i)}(x, h_i) = u^{(i+1)}(x, h_i)\,, \quad v^{(i)}(x, h_i) = v^{(i+1)}(x, h_i) \qquad (i = 1, 2, \ldots, n-1) \end{aligned}\right\} \tag{9}$$

where h_i denotes the height of the interface between the ith and $(i+1)$th layer.

If the temperature distribution $T^{(i)}(x, y)$ is such that its second partial derivative with respect to x contains only a finite number of discontinuities in each layer, then it is possible to represent formally the temperature on the right-hand side of (5) in a single Fourier series. Let

$$T^{(i)}(x, y) = \sum_{m=1}^{\infty} Y_m^{(i)}(y) \sin\frac{m\pi x}{l} \tag{10}$$

where

$$Y_m^{(i)}(y) = \frac{2}{l}\int_0^l T^{(i)}(x, y) \sin\frac{m\pi x}{l}\, dx\,. \tag{11}$$

End conditions (7) are satisfied by expressing the stress function in the form

$$\phi^{(i)} = \sum_{m=1}^{\infty} f_m^{(i)}(y) \sin\frac{m\pi x}{l}\,. \tag{12}$$

Substitution of (10) and (12) into (5) and assuming homogeneous layer yields

$$\frac{1}{E_x^{(i)}}\frac{d^4}{dy^4}f_m^{(i)}(y)-\left(\frac{1}{G_{xy}^{(i)}}-2\frac{\nu_{yx}^{(i)}}{E_y^{(i)}}\right)\left(\frac{m\pi}{l}\right)^2\frac{d^2}{dy^2}f_m^{(i)}(y)$$

$$+\frac{1}{E_y^{(i)}}\left(\frac{m\pi}{l}\right)^4 f_m^{(i)}(y)=-\alpha_x^{(i)}\frac{d^2}{dy^2}Y_m^{(i)}(y)+\alpha_y^{(i)}\left(\frac{m\pi}{l}\right)^2 Y_m^{(i)}(y)\,. \quad (13)$$

Assuming distinct roots, the general solution of the fourth order ordinary differential equation (13) is

$$f_m^{(i)}(y)=A_m^{(i)}\exp(\lambda_{m_1}^{(i)}y)+B_m^{(i)}\exp(\lambda_{m_2}^{(i)}y)$$
$$+C_m^{(i)}\exp(\lambda_{m_3}^{(i)}y)+D_m^{(i)}\exp(\lambda_{m_4}^{(i)}y)+\bar{f}_m^{(i)}(y) \quad (14)$$

where $A_m^{(i)}$, $B_m^{(i)}$, $C_m^{(i)}$, $D_m^{(i)}$ are arbitrary constants, $\bar{f}_m^{(i)}(y)$ is a particular integral of (13), and the four roots are determined as

$$\lambda_{m1,2,3,4}^{(i)}=\pm\frac{m\pi}{l}\left[\frac{\left(\frac{1}{G_{xy}^{(i)}}-2\frac{\nu_{yx}^{(i)}}{E_y^{(i)}}\right)\pm\left\{\left(\frac{1}{G_{xy}^{(i)}}-2\frac{\nu_{yx}^{(i)}}{E_y^{(i)}}\right)^2-\frac{4}{E_x^{(i)}E_y^{(i)}}\right\}^{\frac{1}{2}}}{\frac{2}{E_x^{(i)}}}\right]^{\frac{1}{2}}. \quad (15)$$

In the case of isotropic layers, Equation (13) becomes

$$\frac{d^4}{dy^4}f_m^{(i)}(y)-2\left(\frac{m\pi}{l}\right)^2\frac{d^2}{dy^2}f_m^{(i)}(y)+\left(\frac{m\pi}{l}\right)^4 f_m^{(i)}(y)=-\alpha E\left\{\frac{d^2}{dy^2}Y_m^{(i)}(y)-\left(\frac{m\pi}{l}\right)^2 Y_m^{(i)}(y)\right\} \quad (16)$$

for which the general solution is

$$f_m^{(i)}(y)=(A_m^{(i)}+C_m^{(i)}y)\exp\left(\frac{m\pi}{l}y\right)+(B_m^{(i)}+D_m^{(i)}y)\exp\left(-\frac{m\pi}{l}y\right)+\bar{f}_m^{(i)}(y)\,. \quad (17)$$

Using the general orthotropic results (14) and (15) the stress and displacement (aside from rigid-body terms) components can be written as follows

$$\sigma_x^{(i)}=\sum_{m=1}^{\infty}\sin\frac{m\pi x}{l}\Big[A_m^{(i)}\lambda_{m_1}^{(i)2}\exp(\lambda_{m_1}^{(i)}y)$$
$$+B_m^{(i)}\lambda_{m_2}^{(i)2}\exp(\lambda_{m_2}^{(i)}y)+C_m^{(i)}\lambda_{m_3}^{(i)2}\exp(\lambda_{m_3}^{(i)}y)$$
$$+D_m^{(i)}\lambda_{m_4}^{(i)2}\exp(\lambda_{m_4}^{(i)}y)+\frac{d^2}{dy^2}\bar{f}_m^{(i)}(y)\Big]. \quad (18)$$

$$\sigma_y^{(i)}=-\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2\sin\frac{m\pi x}{l}\,[A_m^{(i)}\exp(\lambda_{m_1}^{(i)}y)$$
$$+B_m^{(i)}\exp(\lambda_{m_2}^{(i)}y)+C_m^{(i)}\exp(\lambda_{m_3}^{(i)}y)$$
$$+D_m^{(i)}\exp(\lambda_{m_4}^{(i)}y)+\bar{f}_m^{(i)}(y)] \quad (19)$$

$$\tau_{xy}^{(i)}=-\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)\cos\frac{m\pi x}{l}\Big[A_m^{(i)}\lambda_{m_1}^{(i)}\exp(\lambda_{m_1}^{(i)}y)$$
$$+B_m^{(i)}\lambda_{m_2}^{(i)}\exp(\lambda_{m_2}^{(i)}y)+C_m^{(i)}\lambda_{m_3}^{(i)}\exp(\lambda_{m_3}^{(i)}y)$$
$$+D_m^{(i)}\lambda_{m_4}^{(i)}\exp(\lambda_{m_4}^{(i)}y)+\frac{d}{dy}\bar{f}_m^{(i)}(y)\Big]. \quad (20)$$

$$u^{(i)} = -\sum_{m=1}^{\infty}\left(\frac{l}{m\pi}\right)\cos\frac{m\pi x}{l}\left\{A_m^{(i)}\left[\frac{\lambda_{m_1}^{(i)2}}{E_x^{(i)}} - \frac{\nu_{xy}^{(i)}\left(\frac{m\pi}{l}\right)^2}{E_y^{(i)}}\right]\exp(\lambda_{m_1}^{(i)}y)\right.$$

$$+B_m^{(i)}\left[\frac{\lambda_{m_2}^{(i)2}}{E_x^{(i)}} - \frac{\nu_{yx}^{(i)}\left(\frac{m\pi}{l}\right)^2}{E_y^{(i)}}\right]\exp(\lambda_{m_2}^{(i)}y)$$

$$+C_m^{(i)}\left[\frac{\lambda_{m_3}^{(i)2}}{E_x^{(i)}} - \frac{\nu_{yx}^{(i)}\left(\frac{m\pi}{l}\right)^2}{E_y^{(i)}}\right]\exp(\lambda_{m_3}^{(i)}y)$$

$$+D_m^{(i)}\left[\frac{\lambda_{m_4}^{(i)2}}{E_x^{(i)}} - \frac{\nu_{yx}^{(i)}\left(\frac{m\pi}{l}\right)^2}{E_y^{(i)}}\right]\exp(\lambda_{m_4}^{(i)}y)$$

$$\left.+\frac{1}{E_x^{(i)}}\frac{d^2}{dy^2}\bar{f}_m^{(i)}(y) - \frac{\nu_{yx}^{(i)}}{E_y^{(i)}}f_m^{(i)}(y) + \alpha_x^{(i)}\,Y_m^{(i)}(y)\right\} \tag{21}$$

$$v^{(i)} = -\sum_{m=1}^{\infty}\sin\frac{m\pi x}{l}\left\{A_m^{(i)}\left[\frac{\nu_{yx}^{(i)}}{E_x^{(i)}}\lambda_{m_1}^{(i)} + \frac{\left(\frac{m\pi}{l}\right)^2}{\lambda_{m_1}^{(i)}E_y^{(i)}}\right]\exp(\lambda_{m_1}^{(i)}y)\right.$$

$$+B_m^{(i)}\left[\frac{\nu_{yx}^{(i)}}{E_x^{(i)}}\lambda_{m_2}^{(i)} + \frac{\left(\frac{m\pi}{l}\right)^2}{\lambda_{m_2}^{(i)}E_y^{(i)}}\right]\exp(\lambda_{m_2}^{(i)}y)$$

$$+C_m^{(i)}\left[\frac{\nu_{yx}^{(i)}}{E_x^{(i)}}\lambda_{m_3}^{(i)} + \frac{\left(\frac{m\pi}{l}\right)^2}{\lambda_{m_3}^{(i)}E_y^{(i)}}\right]\exp(\lambda_{m_3}^{(i)}y)$$

$$+D_m^{(i)}\left[\frac{\nu_{yx}^{(i)}}{E_x^{(i)}}\lambda_{m_4}^{(i)} + \frac{\left(\frac{m\pi}{l}\right)^2}{\lambda_{m_4}^{(i)}E_y^{(i)}}\right]\exp(\lambda_{m_4}^{(i)}y)$$

$$\left.+\frac{\nu_{yx}^{(i)}}{E_y^{(i)}}\frac{d}{dy}f_m^{(i)}(y) + \frac{1}{E_y^{(i)}}\int\bar{f}_m^{(i)}(y)dy - \alpha_y^{(i)}\int Y_m^{(i)}(y)dy\right\} \tag{22}$$

where the integrals represent indefinite integrals.

For each expansion term m are four constants $A_m^{(i)}$, $B_m^{(i)}$, $C_m^{(i)}$, $D_m^{(i)}$ corresponding to the ith layer. If there are n layers in the beam, then the $4n$ constants are determined by the $4n$ Equations (6) and (9) using the results (18)–(22).

3. Numerical examples

Example 1:
As an example let us consider a problem in transient thermal stress. Consider a thin composite beam which is initially under an arbitrary temperature $T = g(x)$. Suppose the beam is insulated on all its faces except at the ends $x = 0$, l which are kept at zero temperature. Also, let us assume that the heat conduction behavior of the composite beam is isotropic and homogeneous. The diffusion equation then gives for any time t the temperature distribution

$$T(x, t) = \sum_{m=1,3,5\ldots}^{\infty} Y_m \exp\left(-\frac{m^2\pi^2\kappa t}{l^2}\right)\sin\frac{m\pi x}{l} \tag{23}$$

where κ is the thermal diffusivity and

$$Y_m = \frac{2}{l}\int_0^l g(x)\sin\frac{m\pi x}{l}\,dx\,. \tag{24}$$

If, for definiteness, we choose

$$g(x) = \frac{T_0}{l^3}(x^3 - l^2 x) \tag{25}$$

then

$$Y_m = \frac{12T_0(-1)^m}{m^3\pi^3}\,,\quad \bar{f}_m^{(i)}(y) = \frac{12T_0(-1)^m l^2}{m^5\pi^5}\,E_y^{(i)}\alpha_y^{(i)}\exp\left(-\frac{m^2\pi^2\kappa t}{l^2}\right). \tag{26}$$

Consider a symmetric 3-layer composite beam which has a very severe difference in the thermal expansion coefficients of the layers. The layers of equal thickness possess the following properties

$$\begin{aligned}
&E_y^{(1)} = E_y^{(3)} = E = 3\times 10^6 \text{ psi}\,,\quad E_x^{(1)} = E_x^{(3)} = 10E\,,\\
&\nu_{xy}^{(1)} = \nu_{xy}^{(3)} = \nu = 0.2\,,\quad \nu_y^{(1)} = \nu_{yx}^{(3)} = \nu/10\,,\\
&G_{xy}^{(1)} = G_{xy}^{(3)} = E/3\,,\quad E_x^{(2)} = E_y^{(2)} = E\,,\quad \nu_{xy}^{(2)} = \nu_{yx}^{(2)} = \nu\,,\quad G_{xy}^{(2)} = E/3\\
&\alpha_y^{(1)} = \alpha_y^{(3)} = \alpha = 2.5\times 10^{-6}\text{ in/in/}^\circ\text{F}\,,\quad \alpha_x^{(1)} = \alpha_x^{(3)} = 6\alpha\,,\quad \alpha_x^{(2)} = \alpha_y^{(2)} = \alpha\,.
\end{aligned}$$

Note that the center layer would be isotropic if $G_{xy}^{(2)} = E/2.4$ and that the outer layers simulate a high modulus unidirectional boron/epoxy lamina.

From (18)–(22) the stress and displacement components can be expressed in terms of 12 arbitrary constants. These constants are determined by the 12 boundary and interface conditions from (6) and (9). The normal stress components in each layer at the mid-span of the beam are given in Table 1. Nine terms of the series were used in the calculation.

Example 1a:
In the case of an isotropic homogeneous beam in Example 1 the stress function is obtained in the form

$$\begin{aligned}
\phi(x, y, t) = \sum_{m=1,3,5}^{\infty} \frac{12T_0(-1)^m l^2 E\alpha}{m^5\pi^5}\Bigg\{1 - \frac{\left(1-\cosh\dfrac{m\pi h}{l}\right)}{\left(\dfrac{m\pi h}{l} + \sinh\dfrac{m\pi h}{l}\right)}\sinh\frac{m\pi y}{l}\\
- \left[1 - \frac{\left(1-\cosh\dfrac{m\pi h}{l}\right)}{\left(\dfrac{m\pi h}{l} + \sinh\dfrac{m\pi h}{l}\right)}\left(\frac{m\pi y}{l}\right)\right]\cosh\frac{m\pi y}{l}\\
- \frac{\sinh\dfrac{m\pi h}{l}}{\left(\dfrac{m\pi h}{l} + \sinh\dfrac{m\pi h}{l}\right)}\left(\frac{m\pi y}{l}\right)\sinh\frac{m\pi y}{l}\Bigg\}\sin\frac{m\pi x}{l}\exp\left(-\frac{m^2\pi^2\kappa t}{l^2}\right).
\end{aligned} \tag{27}$$

The corresponding series for the stress components are identical to the results given in Equations (57) to (61) of Reference [1] which were obtained by reducing a double Fourier series solution to a single Fourier series by explicit summation over one index. This reduction is possible only in certain special cases. The results for an isotropic homogeneous beam are given, in parenthesis, in Table 1 for comparison with the composite beam results.

TABLE 1

y/h	Layer number	$\sigma_x/(T_0 E\alpha)$	$\sigma_y/(T_0 E\alpha 10^{-2})$
0		0.3131 (−0.001166)	0 (0)
1/15		0.3183 (−0.000730)	0.0069 (0.000022)
2/15		0.3269 (−0.000356)	0.0277 (0.000076)
3/15	1	0.3392 (−0.000046)	0.0630 (0.000146)
4/15		0.3552 (0.000202)	0.1131 (0.000219)
5/15		0.3752 (0.000388)	0.1788 (0.000283)
5/15		−0.6746 (0.000388)	0.1788 (0.000283)
6/15		−0.6730 (0.000512)	0.2378 (0.000330)
7/15	2	−0.6721 (0.000574)	0.2673 (0.000355)
8/15		−0.6721 (0.000574)	0.2673 (0.000355)
9/15		−0.6730 (0.000512)	0.2378 (0.000300)
10/15		−0.6746 (0.000388)	0.1788 (0.000283)
10/15		0.3752 (0.000388)	0.1788 (0.000283)
11/15		0.3552 (0.000202)	0.1131 (0.000219)
12/15		0.3392 (−0.000046)	0.0630 (0.000146)
13/15	3	0.3269 (−0.000356)	0.0277 (0.000076)
14/15		0.3183 (−0.000730)	0.0069 (0.000022)
15/15		0.3131 (−0.001166)	0 (0)

Note: $x/l = 0.50$, $h/l = 0.1$, $\kappa t\pi^2/l^2 = 1$.
Nine terms of the series were used in the calculation.

Example 2:
In the next example, let us consider a composite beam with the same properties as in Example 1 under a steady temperature distribution $T(x, y)$ which is an explicit function of x and y. The beam is kept at zero temperature on the three faces $x=0$, l and $y=0$ and at $T(x, h) = (T_0/l^3)(x^3 - l^2 x)$ on face $y=h$. Assuming the temperature distribution $T(x, y)$ in the beam is determined by the harmonic equation, we have

$$T(x, y) = \sum_{m=1,3,5}^{\infty} Y_m(y) \sin \frac{m\pi x}{l} \tag{28}$$

where

$$Y_m(y) = \frac{12T_0(-1)^m}{m^3 \pi^3} \frac{\sinh \dfrac{m\pi y}{l}}{\sinh \dfrac{m\pi h}{l}}. \tag{29}$$

Thus, the particular integral of (13) is given as

$$\bar{f}_m^{(i)}(y) = \frac{12T_0(-1)^m l^2}{m^5 \pi^5} \left(\frac{-\alpha_x^{(i)} + \alpha_y^{(i)}}{\dfrac{1}{E_x^{(i)}} + \dfrac{1+2\nu_{yx}^{(i)}}{E_y^{(i)}} - \dfrac{1}{G_{xy}^{(i)}}} \right) \frac{\sinh \dfrac{m\pi y}{l}}{\sinh \dfrac{m\pi h}{l}}. \tag{30}$$

Again, from (18)–(22) the stress and displacement components can be expressed in terms of 12 arbitrary constants which are determined by 12 boundary and interface conditions from (6) and (9). The normal stress components in each layer at the mid-span of the beam are given in Table 2. This example demonstrates that the proposed single Fourier series method can be applied to heterogeneous anisotropic beams with an arbitrary variation of temperature in the y-direction as well as the x-direction.

TABLE 2

y/h	Layer number	$\sigma_x/(T_0 E\alpha)$	$\sigma_y/(T_0 E\alpha 10^{-2})$
0		1.4077	0
1/15		1.3761	0.0241
2/15		1.3604	0.0968
3/15	1	1.3621	0.2186
4/15		1.3832	0.3902
5/15		1.4245	0.6120
5/15		−1.8904	0.6120
6/15		−2.2941	0.8226
7/15		−2.7000	0.9506
8/15	2	−3.1084	0.9815
9/15		−3.5197	0.9006
10/15		−3.9343	0.6934
10/15		1.4879	0.6934
11/15		1.4655	0.4451
12/15		1.4752	0.2509
13/15	3	1.5187	0.1115
14/15		1.5975	0.0276
15/15		1.7137	0

Note: $x/l = 0.5$, $h/l = 0.1$.
Twelve terms of the series were used in the calculation of σ_x.
Eighteen terms of the series were used in the calculation of σ_y.

4. Conclusions

This paper presents an exact thermoelasticity solution for thin rectangular beams with any number of orthotropic or isotropic layers. The anisotropy of material properties can be the elastic constants or thermal expansion coefficients. The present method is useful for the thermal stress analysis of high strength laminated beams such as boron/epoxy or graphite/epoxy which have very high anisotropy.

A closed form single Fourier series solution can always be obtained by using the present method. This is the main advantage over the double Fourier series method developed by Wah [1]. The single Fourier series method has obvious numerical advantages over double Fourier series formulations.

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